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PLANE PROBLEM OF ELASTICITY THEORY IN A MULTICONNECTED DOMAIN

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Fredholm integral equations are constructed for unknown normal stresses on the domain boundary in the fundamental problem of plane elasticity theory, and their investigation is given.

1. Formulation of the problem. We shall solve the fundamental plane problem of elasticity theory for external forces given on the boundary when the elastic medium occupies a finite or infinite domain of the plane of the complex variable $z = x + iy$, and the external forces on each of the bounding domains of the closed contours are statically equivalent to zero.

We consider the (connected) domain S^+ bounded by simple closed non-intersecting contours $L_0, L_1, \ldots, L_m$, of which the first encloses all the rest. The symbol L will denote the set of all contours. There may be no external contour L_0, and then the domain S^+ will be infinite (plane with holes).

Let S^- denote the domain complementing S^+ in the whole plane, S_k^- the domain enclosed within L_k $(k = 1, 2, \ldots, m)$, and S_0^+, S_0^- the domains inside and outside of L_0, respectively. Furthermore, let $\mathbf{n}$ be the normal to the line L drawn at any point and external relative to S^+, and $\mathbf{T}$ the direction of the positive tangent to L at the same point. For definiteness, we consider that direction on L positive which keeps the domain S^+ on the right. We assume the line L to have continuous curvature everywhere.

2. On properties of stress functions in a multiconnected domain. Representation of biharmonic functions. To facilitate the presentation let us recall certain known formulas of plane elasticity theory and let us indicate the properties of the stress functions. We borrow all the necessary formulas from the book by N. I. Muskhelishvili [1].

The biharmonic stress function $u(x, y)$ (Airy function) and its first derivatives are representable by the formulas

$$u = \operatorname{Re}\,[\bar{z}\varphi(z) + \chi(z)], \quad \bar{z} = x - iy \tag{2.1}$$

$$\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} = \varphi(z) + z\overline{\varphi'(z)} + \overline{\psi(z)}, \quad \psi(z) = \chi'(z)$$

where φ, ψ, χ are analytic functions of the variable z in a domain occupied by an elastic medium. In the finite domain S^+ they have the form (see [1], Sect. 35)

$$\varphi(z) = z\sum_{k=1}^{m} A_k \ln(z - z_k) + \sum_{k=1}^{m} \gamma_k \ln(z - z_k) + \varphi_*(z) \tag{2.2}$$

$$\chi(z) = z \sum_{k=1}^{m} \gamma_k' \ln(z - z_k) + \sum_{k=1}^{m} \gamma_k'' \ln(z - z_k) + \chi_*(z)$$
$$\psi(z) = \sum_{k=1}^{m} \gamma_k' \ln(z - z_k) + \psi_*(z)$$

where z_k is a certain fixed point within S_k^- $(k = 1, 2, \ldots, m)$, A_k is an arbitrary real number, $\gamma_k, \gamma_k', \gamma_k''$ are arbitrary complex constants, and $\varphi_*, \chi_*, \psi_*$ are holomorphic (single-valued, analytic) functions in S^+.

It is seen that for the Airy function to be single-valued in S^+, it is necessary and sufficient to comply with the conditions

$$\bar{\gamma}_k = \gamma_k', \ \operatorname{Im} \gamma_k'' = 0 \ (k = 1, 2, \ldots, m) \tag{2.3}$$

and this means (see [1], Sect. 33) that the external forces on the contour of each of the holes are statically equivalent to zero.

In the presence of (2.3), for the elastic displacements to be single-valued it is necessary (and sufficient) that

$$A_k = 0, \ \gamma_k = 0 \ (k = 1, 2, \ldots, m)$$

and this is equivalent to the requirement of holomorphicity of $\varphi(z)$ in S^+. These latter conditions, expressed in terms of the function $u(x, y)$, are

$$\int_{L_k} \frac{\partial \Delta u}{\partial n} ds = 0, \quad \int_{L_k} \left(x \frac{\partial \Delta u}{\partial n} - \Delta u \frac{\partial x}{\partial n} \right) ds = 0 \tag{2.4}$$
$$\int_{L_k} \left(y \frac{\partial \Delta u}{\partial n} - \Delta u \frac{\partial y}{\partial n} \right) ds = 0 \quad (k = 1, 2, \ldots, m)$$

Therefore, if the stress function $u(x, y)$ is single-valued in the domain S^+, then for the corresponding elastic displacements to be single-valued, it is necessary and sufficient to comply with the conditions (2.4).

Upon compliance with these conditions, the functions $\varphi(z), \psi(z)$ are holomorphic in S^+, and the function $\chi(z)$ will generally be multivalued (it has the form of the right side in the second relationship of (2.2) for $\gamma_k' = 0, \operatorname{Im} \gamma_k'' = 0$).

The above-mentioned properties of the stress function were apparently first established by Grioli [2] on the basis of the paper [3].

In the case when there is no contour L_0, S^+ is an infinite domain located outside the contours $L_1, L_2, \ldots, L_m$. In this case, under the same assumptions relative to the external forces applied to L_k, and the single-valuedness of the displacements, the nature of the stress functions as well as the potentials φ, χ, ψ remains as before in any finite subdomain of the domain S^+.

However, the functions φ and χ will not generally be holomorphic in the whole domain S^+. It is known that the Kolosov — Muskhelishvili potentials admit of the following representations in the neighborhood of the infinitely distant point (see [1], Sect. 36)

$$\varphi(z) = \varphi_0(z) + \varphi_*(z), \ \chi(z) = \chi_0(z) + \chi_*(z) \tag{2.5}$$
$$\varphi_0(z) = \Gamma z, \ \chi_0(z) = \Gamma' z^2, \ \Gamma' = B' + iC'$$

$$\varphi_*(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots \qquad (2.6)$$
$$\chi_*(z) = \lambda \ln z + A_0 + a_0' z + \frac{a_1'}{z} + \frac{a_2'}{z^2} + \cdots$$

Here Γ, B', C' are real constants characterizing the given stress field at infinity, and λ is a certain real constant. The formulas (2.6) mean that the functions $\varphi_*(z)$, $\psi_*(z)$ are holomorphic outside a circle L_R of sufficiently large radius R with center at the origin, including the infinitely distant point.

Let us present certain fundamental formulas for representations of the biharmonic functions which are derivable by a standard method by using appropriate elementary solutions. A certain difficulty occurs just in the case of the infinite domains when the behavior of the function under consideration must be taken into account in infinitely remote parts of the plane. For the class of functions which we shall discuss later, this behavior is characterized completely by (2.5).

The letters P, P_1 will denote points of the domain S^+ or S^-, and Q, P_0 will denote points on the line L, where Q will usually be the variable of integration. The affixes of these points on the complex plane shall be z, z_1, t, t_0, respectively, where $z = x + iy$, $z_1 = x_1 + iy_1$. The arc abscissas of the points t, t_0 are denoted by s and s_0 and are measured in the direction of the positive tangent $\mathbf{T}$ to the line L on each contour L_k $(k = 0, 1, \ldots, m)$. We shall not introduce new symbols to denote the coordinates of the points Q and P_0 by assuming $t = x + iy$, $t_0 = x_0 + iy_0$, or more accurately, $t = x(s) + iy(s)$ and $t_0 = x(s_0) + iy(s_0)$. To avoid the introduction of new symbols for the functions f of points of the line L we shall also sometimes not distinguish the notation $f(t)$ and $f(Q)$ or $f(t_0)$ and $f(P_0)$.

Writing the Green's identity for the pair of functions $u, \Delta v$, and then for $\Delta u, v$, adding the formulas obtained, and relying upon the elementary solution of the biharmonic equation, we obtain the known representations for a finite domain

$$u(P) = \frac{1}{2\pi} \int_L [L^\circ(u;\ P;\ Q) + l(u;\ P;\ Q)]\, ds, \quad P \in S^+ \qquad (2.7)$$
$$0 = \frac{1}{2\pi} \int_L [L^\circ(u;\ P;\ Q) + l(u;\ P;\ Q)]\, ds, \quad P \in S^-$$

Here $(\partial/\partial n_Q$ is differentiation with respect to the normal direction to L° at the point Q):

$$L^\circ(u;\ P;\ Q) = \frac{1}{4}\left[r^2 \ln \frac{1}{r} \frac{\partial \Delta u}{\partial n_Q} - \Delta u \frac{\partial}{\partial n_Q} r^2 \ln \frac{1}{r}\right]$$
$$l(u;\ P;\ Q) = \left(\ln \frac{1}{r} - 1\right) \frac{\partial u}{\partial n_Q} - u \frac{\partial}{\partial n_Q} \ln \frac{1}{r}, \quad Q \in L, \quad r = |z - t|$$

Applying the Laplace operator to both sides of (2.7), we obtain

$$\Delta u(P) = \frac{1}{2\pi} \int_L l(\Delta u;\ P,\ Q)\, ds, \quad P \in S^+ \qquad (2.8)$$
$$0 = \frac{1}{2\pi} \int_L l(\Delta u;\ P,\ Q)\, ds, \quad P \in S^-$$

The function Δu is evidently harmonic in S^+, and should consequently satisfy the condition

$$\int_L \frac{\partial \Delta u}{\partial n} ds = 0 \tag{2.9}$$

The known relationships

$$u(P) = \frac{1}{2\pi} \int_L l(u;\ P,\ Q)\, ds, \quad P \in S^+ \tag{2.10}$$
$$0 = \frac{1}{2\pi} \int_L l(u;\ P,\ Q)\, ds, \quad P \in S^-$$

follow from (2.7) for a function $u(x, y)$ which is harmonic in S.

Let us examine the case of an infinite domain in greater detail. Let us introduce a domain S_R located outside the contours $L_1, L_2, \ldots, L_m$ and within the circle L_R of radius R with center at the origin; R is taken so large that the circle L_R would enclose all the contours L_k, $k = 1, 2, \ldots, m$, whose set, i.e., total domain boundary, will again be denoted by L. Let us set

$$u - u_0 = u_* \tag{2.11}$$

where u_0 is defined by the first formula in (2.1) and the last three formulas in (2.5)

$$u_0 = \operatorname{Re}\,[\bar{z}\,\varphi_0(z) + \chi_0(z)] = \tag{2.12}$$
$$\rho^2\,[\Gamma + B' \cos 2\vartheta - C' \sin 2\vartheta],\ z = \rho e^{i\vartheta}$$
$$\sigma_x^{(\infty)} = 2(\Gamma - B'),\ \sigma_y^{(\infty)} = 2(\Gamma + B'),\ \tau_{xy}^{(\infty)} = 2C' \tag{2.13}$$

(σ_x, σ_y, τ_{xy} are the stress components). Let us recall that the values of the stress components, i.e., the quantities (2.13), are given at infinity when considering the plane problem for domains containing the infinitely distant point of the plane.

The right side of (2.12) is an Airy function corresponding to the homogeneous stress field caused by the forces (2.13). It is biharmonic in any finite part of the plane and for large $|z|$

$$u_0 = O(\rho^2) \tag{2.14}$$

The relationships (2.7) are evidently valid in S_R for the function $u - u_0$. Therefore

$$u_*(P) = \frac{1}{2\pi} \int_{L+L_R} [L^\circ(u_*;\ P,\ Q) + l(u_*;\ P,\ Q)]\, ds, \quad P \in S_R$$
$$0 = \frac{1}{2\pi} \int_{L+L_R} [L^\circ(u_*;\ P,\ Q) + l(u_*;\ P,\ Q)]\, ds, \quad P \in S_k^-, \quad k = 1, 2, \ldots, m$$

Moreover, analogously to (2.9)

$$\int_{L+L_R} \frac{\partial \Delta u_*}{\partial n} ds = 0 \tag{2.15}$$

Let us consider the function

$$w_0(P) = \frac{1}{2\pi} \int_{L_R} [L^\circ(u_*;\ P,\ Q) + l(u_*;\ P,\ Q)]\, ds$$
$$\Delta w_0(P) = \frac{1}{2\pi} \int_{L_k} l(\Delta u_*;\ P,\ Q)\, ds \tag{2.16}$$

where P is an arbitrary point in any finite part of the plane z. On the basis of the first formula in (2.6), the estimates

$$\Delta u = O(|z|^{-2}), \quad \frac{\partial \Delta u_*}{\partial n} = O(|z|^{-3})$$

are valid for large $|z|$, consequently the integral in (2.16) tends to zero at $R \to \infty$. It hence follows that

$$\Delta w_0 = 0 \text{ everywhere on the plane } z \tag{2.17}$$

On the same basis, in the limit as $R \to \infty$, the equality (2.15) becomes

$$\int_L \frac{\partial \Delta u_*}{\partial n}\, ds = \int_L \frac{\partial \Delta u}{\partial n}\, ds = 0 \tag{2.18}$$

It is almost evident that upon compliance with the condition (2.18) the function

$$\frac{1}{2\pi} \int_L [L^\circ(u;\ P,\ Q) + l(u;\ P,\ Q)]\, ds$$

cannot grow more rapidly than $\rho \ln \rho$ at infinity. Taking (2.14) into account, we hence conclude that for large $|z|$

$$w_0 = o(\rho^2) \tag{2.19}$$

Now on the basis of (2.17) and (2.19)

$$\omega_0(x, y) = \alpha x + \beta y + \gamma$$

follows, where α, β, γ are arbitrary real constants. We shall not turn attention to this trinomial since it does not yield nonzero stresses.

In the long run, the representations

$$u(P) = u_0(P) + \frac{1}{2\pi} \int_L [L^\circ(u;\ P,\ Q) + l(u;\ P,\ Q)]\, ds, \quad P \in S^+ \tag{2.20}$$
$$u_0(P) = \frac{1}{2\pi} \int_L [L^\circ(u; P, Q) + l(u;\ P,\ Q)]\, ds, \quad P \in S_k^-, \quad k = 1, 2, \ldots, m$$

hold for the stress function $u(x,\ y)$ in the infinite domain S^+
Correspondingly

$$\Delta u(P) = \Delta u_0(P) + \frac{1}{2\pi} \int_L l(\Delta u;\ P,\ Q)\, ds, \quad P \in S^+ \tag{2.21}$$
$$\Delta u_0(P) = \frac{1}{2\pi} \int_L l(\Delta u;\ P,\ Q)\, ds, \quad P \in S_k^-, \quad k = 1, 2, \ldots, m$$

Now (2.18) should still be added to the preceding formulas. It is understood that the conditions (2.4) that the displacements be single-valued remain unchanged.

3. **Integral equations of the plane problem.** Under the conditions introduced, the plane problem (the problem of plane strain or the generalized plane state of stress of an elastic body) is to determine the biharmonic stress function $u(x, y)$ in the domain S^+ by means of the boundary conditions

$$\frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} = f_1(t) + if_2(t) + c(t) \text{ on } L \tag{3.1}$$

where f_1, f_2 are functions given on L which are single-valued and continuous on each of the contours L_k comprising L, and subject to the conditions:

$$\int_{L_k} f_1\,dx + f_2\,dy = 0 \quad (k = 1, 2, \ldots, m) \tag{3.2}$$

and $c(t) = c_k = \alpha_k + i\beta_k$ on L_k, where c_k are complex constants to be determined together with the function u. Only one of the constants c_k can be fixed arbitrarily; we shall consider $c(t) = 0$ on L_0. Moreover, it is required from the function u that it satisfy the following additional conditions

$$\int_{L_k} \frac{\partial \Delta u}{\partial n}\,ds = 0 \quad (k = 0, 1, 2, \ldots, m) \tag{3.3}$$

$$\int_{L_k}\left(x\frac{\partial \Delta u}{\partial n} - \Delta u\frac{\partial x}{\partial n}\right)ds = 0, \quad \int_{L_k}\left(y\frac{\partial \Delta u}{\partial n} - \Delta u\frac{\partial y}{\partial n}\right)ds = 0 \tag{3.4}$$

$$(k = 1, 2, \ldots, m)$$

In the case of an infinite domain, the boundary conditions (3.1) are given in the set $L_1, L_2, \ldots, L_m$, and in conformity with this, one of the additional conditions, namely (3.3), drops out for $k = 0$. The condition at infinity

$$u(x, y) = u_0(x, y) + O(|z|) \tag{3.5}$$

should still be appended to these conditions, where u_0 is given by (2.12), and the second term is the right side of the first formula in (2.1) if the corresponding right sides of (2.6) are substituted in place of φ and χ. (More accurately, all the second order partial derivatives of u should be given at infinity, and this will be equivalent to giving the function u itself in the form (3.5)). The constant c_k can here also be given arbitrarily on any one of the contours L_k $(k = 1, 2, \ldots, m)$, be considered zero, say, as we shall do. Therefore, the number of unknown real constants to be determined during the solution of the problem equals $2m$ and $2m - 2$ in the case of the finite and infinite domains, respectively.

Proceeding to the construction of the integral equations for the finite domain S^+, we rewrite condition (3.1) in the form

$$u = g_0(t) + \alpha_k x + \beta_k y + \lambda_k \tag{3.6}$$

$$\frac{\partial u}{\partial n} = h_0(t) + \alpha_k\frac{\partial x}{\partial n} + \beta_k\frac{\partial y}{\partial n}$$

$$g_0(t) = \int_0^s f_1\,dx + f_2\,dy, \quad h_0(t) = f_1\frac{\partial x}{\partial n} + f_2\frac{\partial y}{\partial n} \text{ on } L_k\,(k = 0, 1, \ldots, m)$$

According to the above, it can be considered that $\alpha_0 = \beta_0 = 0$, and we shall neglect the constants λ_k since they play no part in solving the problem.

Let us insert (3.6) in place of u and $\partial u/\partial n$ in the right side of the first equality in (2.7), and let us take into account that the function $\alpha_k x + \beta_k y + \lambda_k$ is everywhere harmonic. Then by using (2.10) we obtain the following integral representation for the solution of the problem formulated (the additive constant λ_0 is omitted in the right side):

$$u(P) = \frac{1}{2\pi}\int L^{\circ}(u; P; Q)\, ds + w(P), \quad P \in S^+ \tag{3.7}$$

$$w(P) = \frac{1}{2\pi}\int \left[\left(\ln\frac{1}{r} - 1\right) h_0(Q) - g_0(Q)\frac{\partial}{\partial n}\ln\frac{1}{r}\right] ds \tag{3.8}$$

The function $w(P)$ (the sum of simple and double layer potentials with known densities) is given on the whole plane. It undergoes a discontinuity when the point P passes through the line L, which is not essential to the subsequent exposition.

Let us pass to the limit $P \to P_0$ $(P_0 \in L)$ in (3.7), and let us differentiate the limit equality twice with respect to the contour arc. (For the differentiation with respect to the arc s to be legitimate, it is necessary that the given functions g_0 and h_0 be subject to definite smootheness conditions which are easily refined). Then using the boundary condition (3.6) for the function $u(P)$ desired, we find (the prime denotes differentiation with respect to s)

$$\frac{1}{2\pi}\int_L \frac{\partial^2}{\partial s_0^2} L^{\circ}(u; P_0, Q)\, ds = g(t_0) + \alpha_k x_0'' + \beta_k y_0'' \text{ on } L_k \tag{3.9}$$

$$(k = 0, 1, 2, \ldots, m; \quad t_0 = x_0 + iy_0)$$

$$g(t_0) = g_0''(t_0) - \partial^2 w(P_0)/\partial s_0^2 \tag{3.10}$$

We also pass to the limit as $P \to P_0$ in the first equality in (2.8) and we use the known formula for the limit values of a double-layer potential. Then

$$\Delta u(P_0) = \frac{1}{\pi}\int_L l(\Delta u; P_0, Q)\, ds, \; P_0 \in L \tag{3.11}$$

where the limit values on the line L for the function Δu are in the left side and values of the integral in the first equality in (2.8) on the same line are in the right side.

The set of equations (3.9), (3.11) yields a system of integral equations in the unknown boundary values of the functions Δu and $\partial\Delta u/\partial n$. After the calculations related to differentiation of the operator L° under the integral sign in (3.9), by using the notation

$$\Delta u(Q) = \sigma(Q), \quad \frac{\partial}{\partial n_Q}\Delta u = \nu(Q) \tag{3.12}$$

we write the system of integral equations in the form

$$\frac{1}{2\pi}\int_L [k_{11}(t_0, t)\nu(t) + k_{12}(t_0, t)\sigma(t)]\, ds = g(t_0) + \alpha_k x_0'' + \beta_k y_0'' \tag{3.13}$$

$$\sigma(t_0) + \frac{1}{2\pi}\int [k_{21}(t_0, t)\nu(t) + k_{22}(t_0, t)\sigma(t)]\, ds = 0$$

$$- 4k_{11}(t_0, t) = 2(\ln r + 1) + \cos 2\alpha(t, t_0) - r(2\ln r + 1)\times$$
$$k(t_0)\sin\alpha(t, t_0)$$
$$- 2k_{12}(t_0, t) = r^{-1}\sin\alpha(t_0, t) - r^{-1}\sin 2\alpha(t, t_0)\cos\alpha(t_0, t) -$$
$$k(t_0)\sin\alpha(t, t_0)\sin\alpha(t_0, t) + k(t_0)\cos[\alpha(t) - \alpha(t_0)]\times$$
$$(\ln r + 1/2)$$
$$k_{21}(t_0, t) = 2(\ln r + 1), \quad k_{22}(t_0, t) = 2\frac{\partial}{\partial n_Q}\ln\frac{1}{r} =$$
$$2r^{-1}\sin\alpha(t_0, t) \quad (r = |t - t_0|)$$

where $k(t)$ is the curvature of the line L at the point t, $\alpha(t)$ is the angle formed by the (positive) tangent to the line L at the point t and the axis Ox, and $\alpha(t_0, t)$ is also an angle formed by the same tangent and the vector t_0t, and measured in the positive direction from this latter. The notation $\alpha(t_0)$ and $\alpha(t, t_0)$ has an analogous meaning (the angles $\alpha(t_0, t)$, and $\alpha(t, t_0)$ are shown graphically with their measurement directions in [4]).

The additional conditions (3.3) and (3.4), which become

$$\int_{L_k} \nu(t)\,ds = 0 \quad (k = 0, 1, \ldots, m) \tag{3.14}$$
$$\int_{L_k} [x\nu(t) + y'\sigma(t)]\,ds = 0, \quad \int_{L_k} [y\nu(t) - x'\sigma(t)]\,ds = 0$$
$$(k = 1, 2, \ldots, m)$$

in the new notation, are appended to (3.13).

Let us mention still another relationship which the functions ν and σ should satisfy. To this end, we calculate the normal derivative of the desired function $u(x, y)$ by means of (3.7) and we integrate it along the line L. Taking into account that the function w is harmonic in S^+, we find

$$\int_L \frac{\partial u}{\partial n}\,ds = \frac{1}{4\pi}\int_L \left[a(t)\frac{\partial \Delta u}{\partial n} - b(t)\,\Delta u\right] ds$$
$$a(t) = \int_L |t - t_0|\left(\ln|t - t_0| + \frac{1}{2}\right)\sin\alpha(t, t_0)\,ds_0$$
$$b(t) = \int_L \left\{\cos[\alpha(t) - \alpha(t_0)]\left(\ln|t - t_0| + 1\right) + \frac{1}{2}\cos[\alpha(t_0, t) + \alpha(t, t_0)]\right\} ds_0$$

Using the boundary conditions (3.1) and the notation (3.12) here, we find

$$\frac{1}{2\pi}\int_L [a(t)\nu(t) - b(t)\sigma(t)]\,ds = G, \quad G = 2\int_L f_2\,dx - f_1\,dy \tag{3.15}$$

We append the relationship (3.15) to the system (3.13) and (3.14) in the case of a finite domain ((3.15) is not required in the case of an infinite domain).

Both equations (3.13) will be inhomogeneous in the case of an infinite domain S^+, where their right sides, f and h, respectively, without terms with the constants α_k, β_k will be

$$f(t) = g(t) - d^2u_0 / ds^2, \; h(t) = \Delta u_0 \;\; (\text{ on } L) \tag{3.16}$$

The function u_0 is defined by (2.12). Conditions (3.14) remain unchanged except for one, for $k = 0$, which evidently drops out here.

Even in the case of a finite domain it is understood that a function analogous to u_0 can be introduced if only the plane problem for a (finite) simply-connected domain S_0^+ bounded by the contour L_0 allows of a closed form solution.

It is known that the plane problem of elasticity theory has been studied repeatedly by different authors by the method of integral equations [5 — 9]. The advantage of the integral equations in ν and σ presented above is that their solution affords a possibility of determining the contour stresses of interest in problems for multiconnected domains, without any further calculations.

A somewhat different system of equations for ν and σ is constructed in [10]. One of the equations of this system agrees exactly with the second equation in (3.13).

As in [10], the representation (3.7) taking account of the boundary conditions of the problem in terms of the function w and resulting in the second equation in (3.13), is used above to construct the integral equations. Hence, it is sufficient to use just any one of the two possible conditions of the plane problem to obtain the complete system of integral equations. In this paper, the condition

$$\partial^2 u / \partial s^2 = f_* \text{ on } L$$

is used (f_*, g_* are functions given on L), while the condition

$$\partial^2 u / \partial T^2 = g_* \text{ on } L \tag{3.17}$$

is given in [10], where $\partial / \partial T$ is differentitation with respect to a certain fixed direction coincident with the direction of the tangent at any point of the boundary L.

The meaning of condition (3.17) is not completely clear to this writer; the initial conditions of the problem (3.1) apparently do not permit the determination of the left side of (3.17) along the contour L_k.

It is assumed $g_* \equiv 0$ in condition (3.17) on the outlines of holes free of external forces (only such holes were considered in [10]). Even if it is considered that such conditions are valid, it is not at all convenient to use them. They eliminate the constants α_k, β_k, without which the plane problem is generally incorrect; it is well known that only a suitable selection of unknown constants α, β can assure the necessary single-valuedness of the elastic displacements. The lack of these constants in [10] results in a certain number of linear algebraic equations for discrete values of the solution of the integral equations turning out to be excess.

However, it can happen in some particular cases that a part of the constants α, β or even all the constants are zero. The acceptability of the numerical results in the examples of two circular holes considered in [10] should apparently be explained by this (the constants α, β are generally not needed in a problem with one hole).

Furthermore, the first condition in (3.14) for $k = 0$ is absent in [10], and as will be seen later, this condition plays an essential part in the investigation of the integral equations (the authors of [10] were not concerned with an investigation of their integral equations).

4. Investigation of the integral equations. Let us start with the case of a finite domain and let us consider the homogeneous system of equations

$$\frac{1}{2\pi}\int_L [k_{11}(t_0,t)\nu(t)+k_{12}(t_0,t)\sigma(t)]\,ds=0 \tag{4.1}$$

$$\sigma(t_0)+\frac{1}{2\pi}\int_L [k_{21}(t_0,t)\nu(t)+k_{22}(t_0,t)\sigma(t)]\,ds=0$$

$$\int_{L_k}\nu(t)\,ds=0 \qquad (k=0,1,2,\ldots,m)$$

$$\frac{1}{2\pi}\int_L [a(t)\nu(t)-b(t)\sigma(t)]\,ds=0$$

Let us introduce the function v biharmonic in S^+, defined for arbitrary continuous functions ν and σ on L by the integral

$$v(P)=\frac{1}{2\pi}\int_L\left[r^2\ln\frac{1}{r}\nu(Q)-\sigma(Q)\frac{\partial}{\partial n}r^2\ln\frac{1}{r}\right]ds \tag{4.2}$$

Let us call it v_0 for some solution ν_0, σ_0 of the homogeneous system, and let us apply the Laplace operator to it. We obtain

$$\Delta v_0(P)=\frac{1}{2\pi}\int_L\left[\left(\ln\frac{1}{r}-1\right)\nu_0(Q)-\sigma_0(Q)\frac{\partial}{\partial n}\ln\frac{1}{r}\right]ds \tag{4.3}$$

The limit values of the right side of (4.3) on L are zero when the point P tends from S^- to the point P_0 on L according to the first equation in (4.1). But the right side of (4.3) is harmonic in any of the domains S_k^- $(k=0,1,\ldots,m)$ and is bounded at infinity because of the third equation in (4.1). Hence, on the basis of uniqueness of the harmonic function

$$\frac{1}{2\pi}\int_L\left[\left(\ln\frac{1}{r}-1\right)\nu_0(Q)-\sigma_0(Q)\frac{\partial}{\partial n}\ln\frac{1}{r}\right]ds=0,\quad P\in S^- \tag{4.4}$$

After passing to the limit as $P\to P_0$ from inside and outside of S^+, we obtain, respectively, from (4.3) and (4.4)

$$\Delta v_0(P_0)=\sigma_0(P_0)\ \text{ on } L \tag{4.5}$$

Let us fix the point P_0 on L, and let us differentiate (4.3) and (4.4) with respect to the direction of the external normal to L at the point P_0.

Passage to the limit in the equalities obtained in this manner as $P_0\to P$, we find

$$\frac{\partial}{\partial n_0}\Delta v_0(P_0)=\nu_0(P_0)\ \text{ on } L \tag{4.6}$$

and (4.2), whose left side equals v_0 for $\nu\to\nu_0, \sigma=\sigma_0$ becomes

$$v_0(P)=\frac{1}{2\pi}\int_L L^\circ(v_0;P,Q)\,ds,\qquad P\in S^+ \tag{4.7}$$

There hence follows on the basis of the Green's formula (2.7)

$$\frac{1}{2\pi}\int_L l(v_0; P, Q)\,ds = 0, \quad P \in S^+ \tag{4.8}$$
$$\frac{1}{2\pi}\int_L [L^\circ(v_0; P, Q) + l(v_0; P, Q)]\,ds = 0, \quad P \in S^-$$

Now, let us note that a biharmonic function in S^+, equal to the right side of (4.7), is continuous up to the boundary L. Its limit values have a second order derivative with respect to the arc of the contour, which equals zero identically on L because of the first equation in (4.1). This means that the function mentioned should take on constant values on L, and the second equality in (4.8) yields

$$\frac{1}{2\pi}\int_L l(v_0; P, Q)\,ds = -\lambda_k, \quad P \in L_k, \quad k = 0, 1, \ldots, m \tag{4.9}$$

Limit values of the appropriate integral from S^- on L are in the left side of (4.9), and λ_k are certain perfectly definite real constants.

On the basis of the known properties of a double-layer potential, $v_0 = \lambda_k$ on L_k $(k = 0,1,\ldots, m)$ follows from the first equality in (4.8) and (4.9). The function expressed by the second integral in the left side of the second equality in (4.8) is harmonic in each of the domains S_k^- comprising S^-, and is bounded at infinity by virtue of the last equation in (4.1). Hence, (4.9) means

$$\frac{1}{2\pi}\int_L l(v_0; P, Q)\,ds = -\lambda_k, \quad P \in S_k^-, \quad k = 0, 1, \ldots, m \tag{4.10}$$

Using the limit properties of a simple-layer potential exactly as in the derivation of (4.6), we find from the first equality in (4.8) and (4.9)

$$\partial v_0 / \partial n = 0 \quad \text{on} \quad L_k \quad (k = 0, 1, \ldots, m)$$

Now let us use the Green's identity for the functions Δv_0, v_0, where v_0 is given in S^+ by (4.7). We have

$$\iint_{S^+} (\Delta v_0)^2\,ds = \int_L \left(\Delta v_0 \frac{\partial v_0}{\partial n} - v_0 \frac{\partial \Delta v_0}{\partial n}\right) ds \tag{4.11}$$

Because of the equations established above for the limit values of the function v_0 and its normal derivative, as well as the relations written by the third equality in (4.1) for $v = v_0$, the right side of the preceding formula equals zero. Therefore, $\Delta v_0 = 0$ in S^+, and $v_0(Q) = \sigma_0(Q) = 0$ on L on the basis of (4.5) and (4.6). Uniqueness of the solution of the system (4.1) is proved.

Turning to the case of an infinite domain, we first clarify the behavior of the function (4.2) for large $|z|$. After elementary calculations, we obtain the asymptotic formula (A and B are real and complex constants)

$$8\pi v(P) = A\,|z^2|\ln|z| + (B\bar{z} + \bar{B}z)(\ln|z| + {}^1\!/_2) + O(\ln|z|) \quad (|z| \to \infty) \tag{4.12}$$

$$-A = \int\limits_L \nu(t)\,ds, \qquad B = \int\limits_L [t\nu(t) - it'\sigma(t)]\,ds \tag{4.13}$$

In the case of the infinite domain, the homogeneous system will consist of the first two equations in (4.1) obtained from the inhomogeneous equations for $f_1 = f_2 = \alpha_k = \beta_k = 0$ on L_k $(k = 1, \ldots, m)$, $\Gamma = B' = C' = 0$ and the following additional equations to a total number of $m + 2$:

$$\int\limits_{L_k} \nu(t)\,ds = 0 \qquad (k = 1, \ldots, m) \tag{4.14}$$

$$\int\limits_L [x\nu(t) + y'\sigma(t)]\,ds = \int\limits_L [y\nu(t) - x'\sigma(t)]\,ds = 0$$

(L is again the set of contours $L_1, L_2, \ldots, L_m$).

Analogously to the preceding, we introduce the function v_0 defined in terms of ν_0, σ_0 the solution of the homogeneous system, by means of (4.2), and the limit relationships (4.5) and (4.6) are established on the basis of the Green's formula (2.21) for $u_0 \equiv 0$ from the second equation in (4.1) and the first equations in (4.14). Using (2.20) for $u_0 \equiv 0$ and the first equation in (4.1) we find as in the case of a finite domain

$$v_0 = -\lambda_k, \quad \partial v_0 / \partial n = 0 \quad \text{on } L_k \ (k = 1, 2, \ldots, m)$$

where λ_k is some real constant.

Formula (4.11) remains for an infinite domain. In order to prove this, the integral

$$\int\limits_{L_k} \left(\Delta v_0 \frac{\partial v_0}{\partial n} - v_0 \frac{\partial \Delta v_0}{\partial n}\right) ds \tag{4.15}$$

is investigated for large $|z|$.

According to (4.14), the coefficients A and B from (4.13), which correspond to the function v_0, equal zero, and (4.12) yields the following estimate for v_0

$$v_0 = O(\ln|z|) \tag{4.16}$$

It shows that the function v_0 is representable outside L_k by (2.1), where φ and χ are the series (2.6) without the coefficients a_0 and a_0'. On the basis of the above, the estimate (4.16) allows the following strengthening:

$$v_0 = O(1), \quad \frac{\partial v_0}{\partial n} = O(|z|^{-1}), \quad \Delta v_0 = O(|z|^{-2}), \quad \frac{\partial \Delta v_0}{\partial n} = O(|z|^{-3})$$

In the presence of the preceding formulas, the integral (4.15) tends to zero as $R \to \infty$ and this indeed proves the validity of (4.11) in the case under consideration. As before, $\Delta v_0 = 0$ in S^+ follows, meaning $\nu_0(Q) = \sigma_0(Q) = 0$ on L.

Let us turn to the inhomogeneous integral equations, and because of the complete analogy we limit ourselves to considering the case of a finite domain. Let us first note that the first equation in (3.14) can be written as

$$\frac{d^2}{ds_0^2}\Omega(P_0) = 0 \tag{4.17}$$

$$\Omega(P_0) = \frac{1}{8\pi}\int\limits_L \left[r^2 \ln\frac{1}{r}\nu(Q) - \sigma(Q)\frac{\partial}{\partial n}r^2\ln\frac{1}{r}\right] ds + w(P_0) - \tag{4.18}$$
$$g_0(P_0) - \alpha_k x_0 - \beta_k y_0 \text{ on } L_k \quad (k = 0, 1, \ldots, m)$$

Limit values of the appropriate functions from the domain S^+ are everywhere in the preceding equality.

The function Ω defined by (4.18), is single-valued and continuous on each of the contours L_k for any ν and σ. Hence, the equation obtained from (4.17) by differentiation with respect to the arc s_0, namely, the equation

$$d^3 / ds_0^3\Omega\ (P_0) = 0 \tag{4.19}$$

is equivalent to the initial equation. (For the differentiation of (4.17) with respect to the arc s_0 to be valid it is first necessary to raise the smoothness of the line L, to require, say, that the coordinates of its points x, y have continuous derivatives with respect to s to the third order). Hence, if the first equation in (3.13) is replaced by (4.19) in the integral equations system under consideration, then we obtain a system completely equivalent to the initial system.

On the other hand, the system (4.19) and the second equation in (3.13) will be a system of singular integral equations of normal type with zero index (see [4], Chapter VI), as can be seen on the basis of (3.13) for the kernel k_{ij}. As is known, the Fredholm theory which we shall use, is valid for such a system.

The presence of the solution of the system (4.19) and the second equation in (3.13) is assured by the existence theorem for the solution of the fundamental biharmonic problem for multiconnected domains [5, 9]. (We prefer not to resort to the appropriate Fredholm theorem here to avoid the investigation of the associated homogeneous system). The final number of arbitrary constants in the solution in terms of nontrivial solutions of the corresponding homogeneous system of integral equations is determined uniquely on the basis of the uniqueness, proved above, for the solution of the system (3.13), the first equation in (3.14) and (3.15). Hence, the solution ν, σ mentioned will depend exclusively on g, α_k, β_k and will contain the unknown constants α_k β_k linearly.

If this solution is substituted into the last two conditions in (3.14), we then obtain a system of linear equations to determine α_k, β_k, which becomes

$$\sum_{k=1}^{2m} a_{ik}\alpha_k = F_i \quad (i = 1, 2, \ldots, 2m) \tag{4.20}$$

in the notation

$$\alpha_j = d_j,\ \beta_j = \alpha_{m+j}\ (j = 1, 2, \ldots, m)$$

where F_i are unknown constants dependent on the functions f_1, f_2 given on L and which vanish for $f_1(t) = f_2(t) = 0$ on L, and a_{ik} are also definite constants dependent only on the geometry of the domain S^+ (there is no real need to solve the integral equations in the system (4.20)).

The system (4.20) defines the constants α_k $(k = 1, 2, \ldots, 2m)$ uniquely. In fact, let the system have the solution α_k° for $F_k = 0$ $(k = 1, 2, \ldots, 2m)$. Then the system of integral equations (3. 13), (3. 14) will have a solution for $g(t) = 0$ on L, and $\alpha_k = \alpha_k^\circ$, $\beta_k = \overset{\circ}{\alpha}_{m+k}$. Substituting this value of v_0, σ_0 in the right side of (4. 2), we construct the functions $v_0(P)$ which is biharmonic in S^+ and satisfies the following boundary conditions on L

$$\frac{\partial v_0}{\partial x} + i\frac{\partial v_0}{\partial y} = \alpha_k^\circ + i\overset{\circ}{\alpha}_{m+k} \text{ on } L_k \; (k = 0, 1, \ldots, m)$$

and also the conditions of single-valuedness of the elastic displacements. Then $v_0 =$ const in S^- and all the $\alpha_k^\circ = 0$ on the basis of the uniqueness of the solution of the plane problem.

Therefore, the system of integral equations (3. 13) — (3. 15) is solvable for any (sufficiently smooth) boundary values f_1, f_2 subjected to the above-mentioned constraints, and determines the functions v, σ and the previously unknown constants α_k, β_k uniquely.

The integral equations (3. 13) have kernels with logarithmic singularities and, as follows from the above, are equivalent in the sense of the existence of the solution of regular equations of the second kind. A numberical realization of the solutions can be accomplished by known, well-developed methods.

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